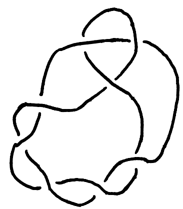
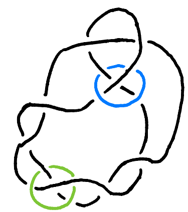
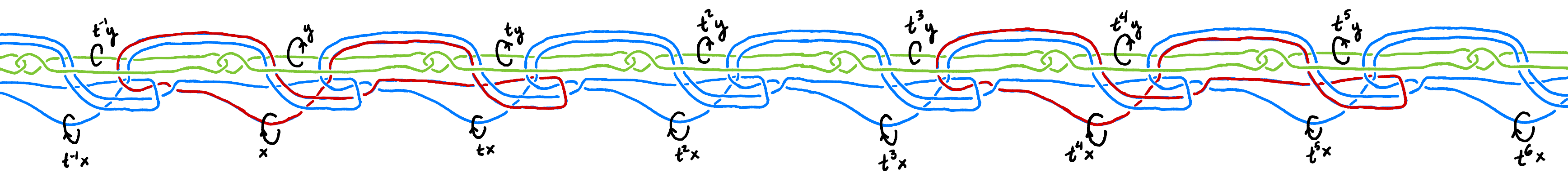
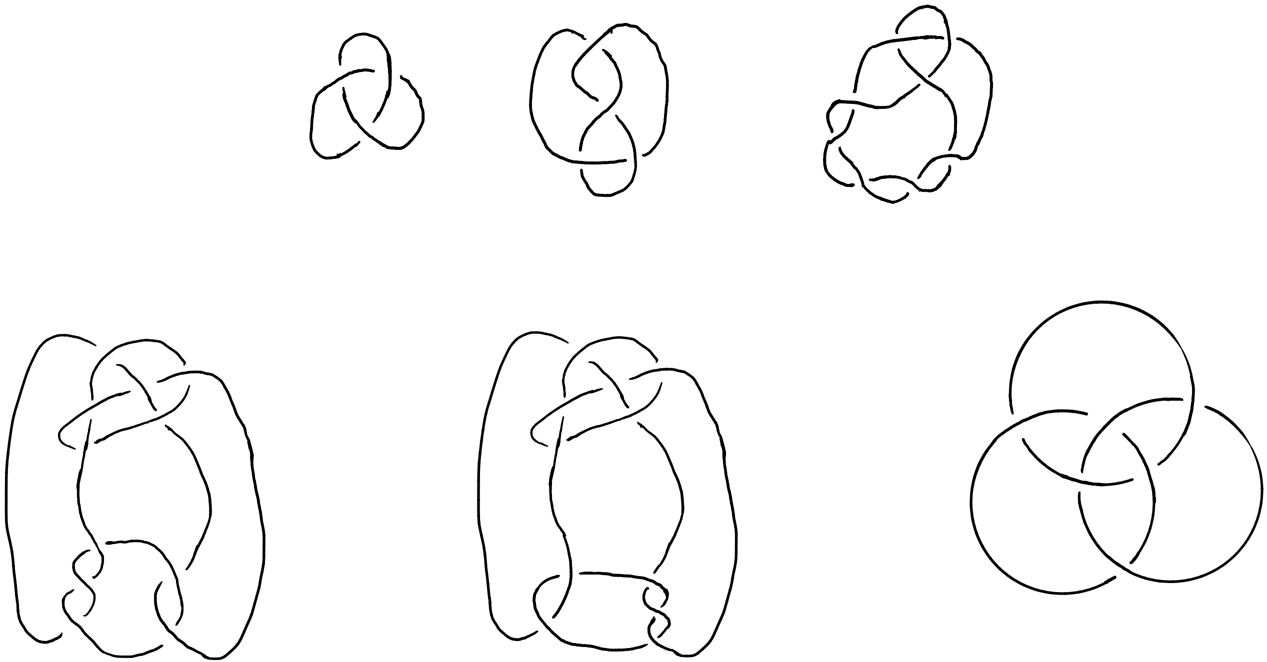
Alexander Polynomials and Gordian Distance





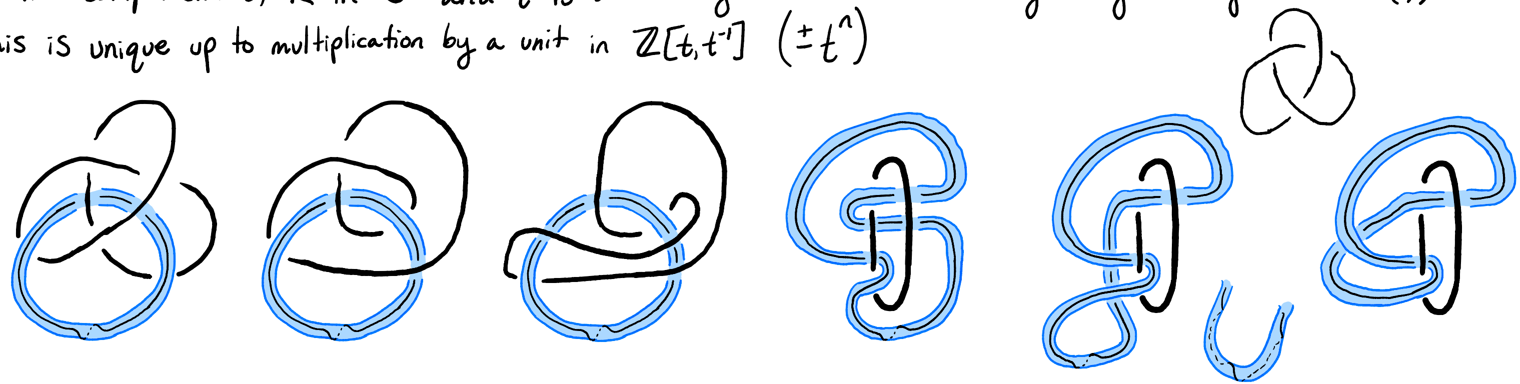


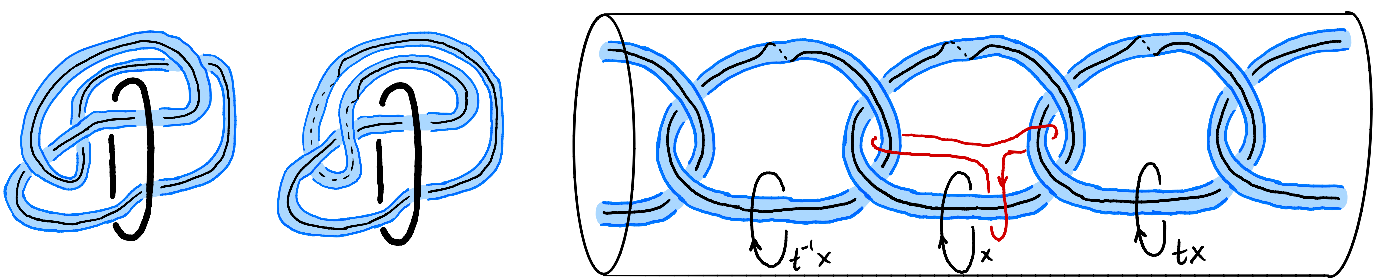
What is knot theory?



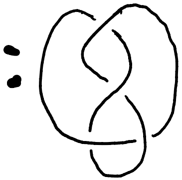
What is the Alexander Polynomial?

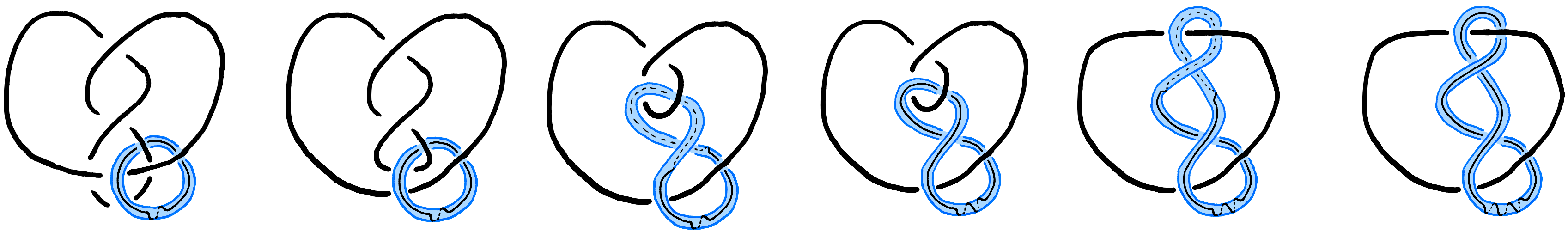
<u>Def (Alexander, 1923)</u> Given a knot K, the Alexander polynomial $\Delta_K(t)$ is the determinant of a presentation matrix (Alexander matrix) for $H_1(X_\infty)$ as a module over $\mathbb{Z}[t,t^{-1}]$ where X_∞ is the infinite cyclic cover of the complement of K in S^3 and t is a covering transformation "moving along the cyclic cover" (?)
This is unique up to multiplication by a unit in $\mathbb{Z}[t,t^{-1}]$ $(\pm t^n)$

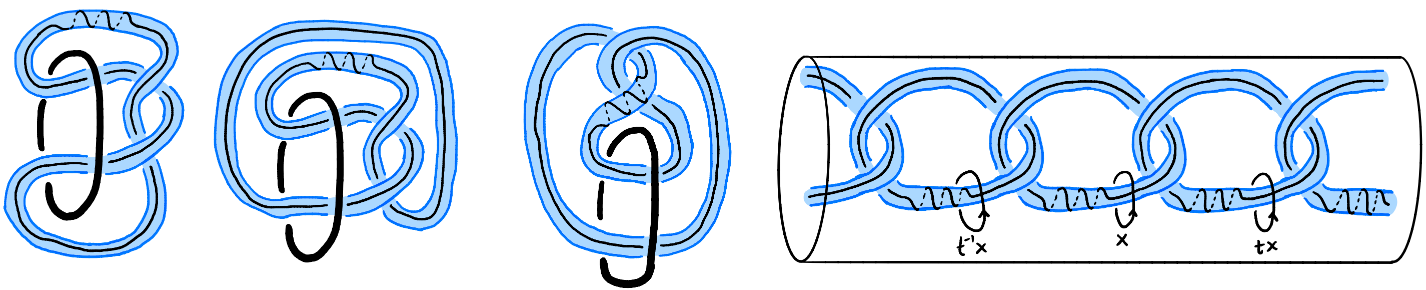




$$\frac{tx - x + t^{-1}x}{[t-1+t^{-1}]}$$

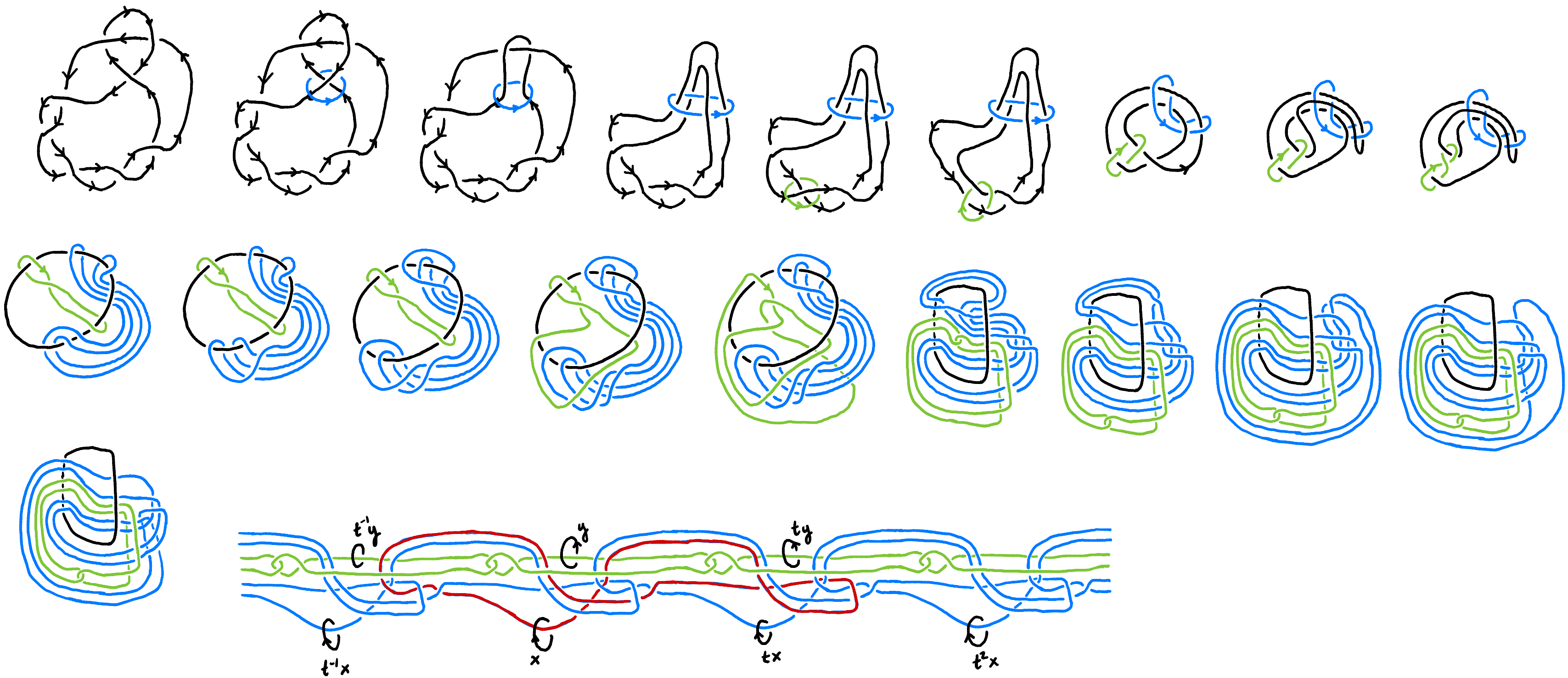
Another example computation: 



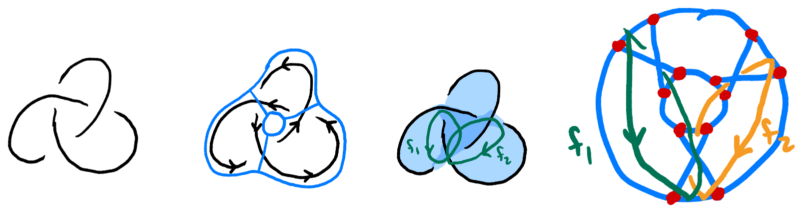


$t^{-1}x - 3x + tx$

Another partial computation:



Another method of computing the Alexander polynomial:



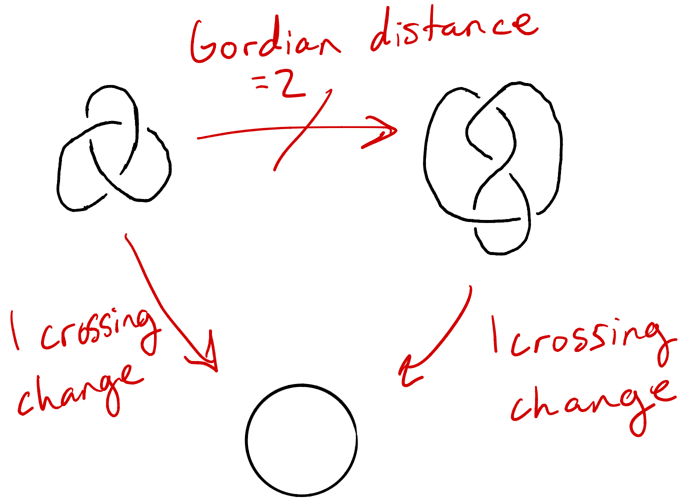
$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$tA - A^T = \begin{bmatrix} -t & -t \\ 0 & -t \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -t+1 & -t \\ 1 & -t+1 \end{bmatrix}$$

$$\Delta 3_1 = \det(tA - A^T) = t^2 - 2t + 1 + t = t^2 - t + 1$$

How can we understand the complexity of knots?

Unknotting number and Gordian distance.



How do these ideas interact?

Thm (Kondo, 1978): For any Alexander polynomial $p(t)$, there exists a knot K with unknotting number 1 such that $\Delta_K(t) = p(t)$.

$p(t)$ Laurant polynomial s.t.
$p(t) = p(t^{-1})$ & $p(1) = \pm 1$

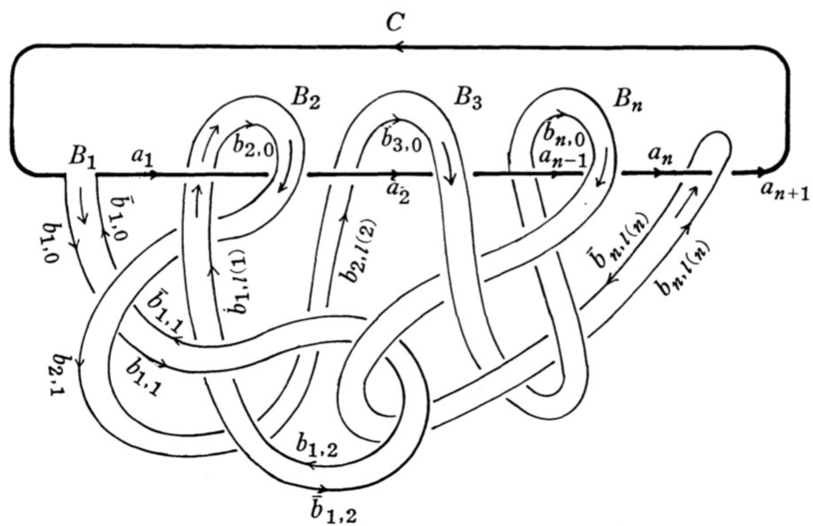


Fig. 4

Question: Does there exist a nontrivial Alexander polynomial $a(t)$ such that for any Alexander polynomial $b(t)$, there exist a pair of knots K_a and K_b with Gordian distance 1 such that $\Delta_{K_a}(t) = a(t)$ and $\Delta_{K_b}(t) = b(t)$?

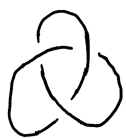
Answer (Kawauchi, 2011): Yes! This is the case for any Alexander polynomial $a(t)$ of slice type (meaning $a(t) = c(t)c(t^{-1})$ for some Laurent polynomial $c(t)$)

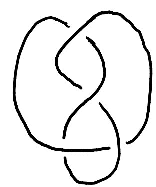
Jong's Problem: Does there exist a pair of Alexander polynomials $a(t)$ & $b(t)$ such that any two knots K_a & K_b where $\Delta_{K_a}(t) = a(t)$ and $\Delta_{K_b}(t) = b(t)$ have Gordian distance at least 2?

Answer (Kawauchi, 2011): Yes! For example:

$a(t) = t - 1 + t^{-1}$ ← Alexander polynomial of trefoil

$b(t) = -t + 3 - t^{-1}$ ← Alexander polynomial of figure 8 knot





Question: Does there exist a nontrivial knot K such that for any Alexander polynomial $a(t)$, there exists some knot K_a such that $\Delta_{K_a}(t) = a(t)$ and the Gordian distance between K and K_a is 1?

Answer: Open

But, I think we can eliminate knots with monic Alexander polynomial from the running using a procedure for characterizing the Alexander polynomials of knots with Gordian distance one by Nakanishi and Okada (2011).

Notation: For any knot K, let $K^\times$ be the knots Gordian distance 1 from K and let $\Delta K^\times$ be the set of Alexander polynomials for knots in $K^\times$. Let ΔK be the set of all Alexander polynomials.

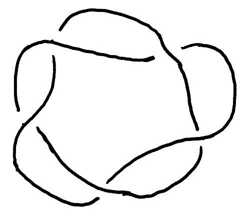
Thm (Nakanishi and Okada, 2011):

$$\Delta 10_{132}^\times \cap \Delta 5_1^\times \neq \emptyset$$

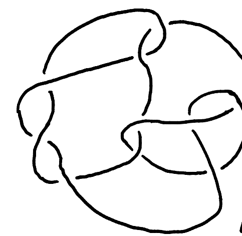
$$\Delta 10_{132}^\times \setminus \Delta 5_1^\times \neq \emptyset$$

$$\Delta 5_1^\times \setminus \Delta 10_{132}^\times \neq \emptyset$$

$$\Delta K \setminus (\Delta 5_1^\times \cup \Delta 10_{132}^\times) \neq \emptyset$$

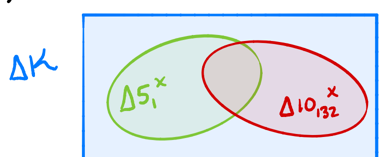


5_1



10_{132}

$$\Delta_{5_1}(t) = t^2 - t + 1 - t^{-1} + t^{-2} = \Delta_{10_{132}}(t)$$



Thank you!